HW3B. ⑨ The reflection $R_2 : \mathbb{R}^3 \to \mathbb{R}^3$ across the plane $P_2 = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y + z = 0\}$;

Method 1: Change of Basis.

$N = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$; Choose $d_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$;

Then, R_2 satisfies $R_2(N) = -N$, $R_2(d_1) = d_1$, and $R_2(d_2) = d_2$.

Equivalently, $A\begin{pmatrix} N & d_1 & d_2 \end{pmatrix} = \begin{pmatrix} -N & d_1 & d_2 \end{pmatrix}$

and $\cancelto{I_3}{A\begin{pmatrix} N & d_1 & d_2 \end{pmatrix}\begin{pmatrix} N & d_1 & d_2 \end{pmatrix}^{-1}} = \begin{pmatrix} -N & d_1 & d_2 \end{pmatrix}\begin{pmatrix} N & d_1 & d_2 \end{pmatrix}^{-1}$;

By Symbolab: $\begin{pmatrix} N & d_1 & d_2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & -2 & -1 \\ 1 & 0 & 3 \end{pmatrix}^{-1} = \frac{1}{42}\begin{pmatrix} 6 & 9 & 3 \\ 10 & -6 & -2 \\ -2 & -3 & 13 \end{pmatrix}$; This is tedious to do by hand.

Then, $A = \begin{pmatrix} -2 & 3 & 0 \\ -3 & -2 & -1 \\ -1 & 0 & 3 \end{pmatrix} \frac{1}{42}\begin{pmatrix} 6 & 9 & 3 \\ 10 & -6 & -2 \\ -2 & -3 & 13 \end{pmatrix} = \frac{1}{42}\begin{pmatrix} 18 & -36 & -12 \\ -36 & -12 & -18 \\ -12 & -18 & 36 \end{pmatrix} = \boxed{\frac{1}{7}\begin{pmatrix} 3 & -6 & -2 \\ -6 & -2 & -3 \\ -2 & -3 & 6 \end{pmatrix}}$;

Method 2: By the Householder Formula.

$A = \frac{1}{\langle N, N \rangle}\left(\langle N, N \rangle I_3 - 2NN^T\right)$; $\langle N, N \rangle = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 4 + 9 + 1 = 14$;

$\langle N, N \rangle I_3 - 2NN^T = 14\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} - 2\begin{pmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} + \begin{pmatrix} -8 & -12 & -4 \\ -12 & -18 & -6 \\ -4 & -6 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -12 & -4 \\ -12 & -4 & -6 \\ -4 & -6 & 12 \end{pmatrix}$;

$A = \frac{1}{14}\begin{pmatrix} 6 & -12 & -4 \\ -12 & -4 & -6 \\ -4 & -6 & 12 \end{pmatrix} = \boxed{\frac{1}{7}\begin{pmatrix} 3 & -6 & -2 \\ -6 & -2 & -3 \\ -2 & -3 & 6 \end{pmatrix}}$;